MATH2060B Tutorial 6.

Q11 If f is bounded on $[a,b]$ and if the restriction of f to every interval $[c,b]$ where $c\in(a,b)$ is Riemann integrable, show that $f\in\mathcal{R}[a,b]$ and $\int_c^b f \to \int_a^b f$ as $c\to a^+$.

Sol Want to apply Squeeze's Thm, i.e. for each $\varepsilon>0$, construct $\alpha_\varepsilon, \omega_\varepsilon \in \mathcal{R}[a,b]$ s.t. $\alpha_\varepsilon \le f \le \omega_\varepsilon$ and

$$\int_a^b \omega_\varepsilon - \alpha_\varepsilon < \varepsilon.$$

Fix $\varepsilon>0$. Consider $a < c_\varepsilon < \min\{a+\frac{\varepsilon}{2M}, b\}$

where $M>0$ is a bound of f, i.e. $|f(x)|\le M\ \forall x\in[a,b]$.

Define $\alpha_\varepsilon(x) := \begin{cases} -M, & x\in[a,c_\varepsilon) \\ f(x), & x\in[c_\varepsilon,b] \end{cases}$, $\omega_\varepsilon(x) := \begin{cases} M, & x\in[a,c_\varepsilon) \\ f(x), & x\in[c_\varepsilon,b] \end{cases}$

so $\forall x\in[a,b]$, $\alpha_\varepsilon(x) \le f(x) \le \omega_\varepsilon(x)$.

Because restrictions of α_ε and ω_ε

$\to$ to $[a,c_\varepsilon]$ is a step function

$\to$ to $[c_\varepsilon,b]$ is $f|_{[c_\varepsilon,b]}$

both integrable.

by assumption $\because c_\varepsilon\in(a,b)$

By additivity thm, $\alpha_\varepsilon, \omega_\varepsilon \in \mathcal{R}[a,b]$.

Since $\omega_\varepsilon(x) - \alpha_\varepsilon(x) = \begin{cases} 2M, & x\in[a,c_\varepsilon) \\ 0, & x\in[c_\varepsilon,b] \end{cases}$.

Thm 7.2.5 $\Rightarrow \int_a^b (\omega_\varepsilon - \alpha_\varepsilon) = 2M\cdot(c_\varepsilon - a) < 2M\cdot\frac{\varepsilon}{2M} = \varepsilon$.

Squeeze's Thm 7.2.3 implies $f\in\mathcal{R}[a,b]$.

Note: $-M \le f(x) \le M,\ \forall x\in[a,c] \Rightarrow -M(c-a) \le \int_a^c f \le M(c-a)$

Therefore $\left|\int_a^b f - \int_c^b f\right| \overset{\text{additivity Thm.}}{=} \left|\int_a^c f\right| \le M(c-a) \to 0$ as $c\to a^+$

$\Rightarrow \int_c^b f \to \int_a^b f$ as $c\to a^+$.

Q12 Show that $g(x) := \begin{cases} \sin(1/x) & , x \in (0,1] \\ 0 & , x = 0 \end{cases}$

belongs to $\mathcal{R}[0,1]$.

Sol (1) $|g(x)| \le 1 \quad \forall x \in [0,1]$

(2) $g|_{[c,1]}$, $\forall\, 0 < c < 1$ is continuous

$\Rightarrow$ integrable on $[c,1]$

Q11 implies that $g \in \mathcal{R}[0,1]$.

Q17 If f and g are continuous on $[a,b]$ and $g(x) > 0$ for all $x \in [a,b]$, show that there exists $c \in [a,b]$ such that

$$\int_a^b fg = f(c) \int_a^b g. \qquad (*)$$

Conditions on g can be weakened: proof only requires g to be ① non-negative ② Riemann integrable.

Pf. Let $m = \inf_{x \in [a,b]} f(x)$, $M = \sup_{x \in [a,b]} f(x)$.

Note: f continuous on $[a,b] \Rightarrow$ there exist $\alpha, \beta \in [a,b]$ such that $m = f(\alpha)$, $M = f(\beta)$

Since $m \le f(x) \le M$ and $g(x) \ge 0 \quad \forall x \in [a,b]$

$$m g(x) \le f(x) g(x) \le M g(x).$$

By Thm 7.1.5, $m \int_a^b g \le \int_a^b fg \le M \int_a^b g$.

Because $g \ge 0$, the same theorem implies $\int_a^b g \ge 0$.

Case 1 $\int_a^b g = 0$, then $\int_a^b fg = 0$.

So $(*)$ holds for any $c \in [a,b]$.

Case 2 $\int_a^b g > 0$, so $\underset{\underset{f(\alpha)}{\parallel}}{m} \le \dfrac{\int_a^b fg}{\int_a^b g} \le \underset{\underset{f(\beta)}{\parallel}}{M}$.

Intermediate value thm $\Rightarrow \exists\, c \in [\alpha, \beta] \subseteq [a,b]$ such that $f(c) = \int_a^b fg \Big/ \int_a^b g$, i.e. $(*)$ holds. $\square$

$\rightarrow$ Note that if we replace g by $-g$, then we see that same conclusion holds when $g \in \mathcal{R}[a,b]$ and g non-positive. (and f still assumed continuous).

$\rightarrow$ "Counterexample" if g changes sign on $[a,b]$:

consider $f(x) = g(x) = x$ on $[-1,1]$, then

$$0 < \int_{-1}^{1} x^2 dx \neq f(c) \int_{-1}^{1} x\, dx = 0 \quad \forall c \in [-1,1].$$

Q8 Suppose f is continuous on $[a,b]$, $f(x) \geq 0$ $\forall x \in [a,b]$ and $\int_a^b f = 0$. Prove that $f(x) = 0$ $\forall x \in [a,b]$.

Pf Suppose for a contradiction $f(c) > 0$ for some $c \in (a,b)$.

Then by continuity of f at c (with $\varepsilon = \frac{f(c)}{2} > 0$), there exists $\delta > 0$ such that $(c-\delta, c+\delta) \subset [a,b]$

and $x \in (c-\delta, c+\delta) \Rightarrow |f(x) - f(c)| < \frac{f(c)}{2}$

$$\Rightarrow f(x) > \frac{f(c)}{2} > 0.$$

Therefore,

$$\int_a^b f = \int_a^{c-\delta} f + \int_{c-\delta}^{c+\delta} f + \int_{c+\delta}^{b} f \quad \text{(additivity)}$$

$$\geq \int_{c-\delta}^{c+\delta} f \quad (\text{Thm } 7.15c) \quad \because f \geq 0$$

$$\geq \int_{c-\delta}^{c+\delta} \frac{f(c)}{2} = \delta f(c) > 0$$

So this is a contradiction. Thus $f(x) = 0$ $\forall x \in (a,b)$.

By continuity of f at a, b, we must have $f = 0$ on $[a,b]$.